and $\int_0^\infty dt$. Since we shall be concerned below with the fact that certain other limits cannot be interchanged without affecting the results, a comment about this particular interchange seems necessary. We shall assume that $\vec{k}$ space is finite (say, one Brillouin zone) and that the functions we consider like $e^{Dt}D'f^s$ are continuous in $\vec{k}$. Under these assumptions, the interchange of $\int d^3k$ with the limit $\eta \to 0+$ and $\int_0^\infty dt$ is justifiable.

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Hot-Electron Problem. II. Two Soluble Models

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Two exactly soluble models for the hot-electron problem are introduced. The models are used to illustrate and gain insight into various approaches to the time-dependent and steady-state problems and to the perturbation theory from a known steady state. For one of the models it is shown that the optimum choice for a generalized self-scattering rate violates Rees's condition (that the self-scattering rate be positive) everywhere.

I. INTRODUCTION

In the preceding paper,[1] we have presented an operator formulation of the hot-electron problem that is developed in analogy with the operator formulation of quantum mechanics. The analogy rests on the fact that both the Boltzmann equation and the time-dependent Schrödinger equation are first order in $\partial/\partial t$.

In the present paper we wish to gain further insight into hot-electron problems and mathematical techniques for their solution by introducing two soluble models. We shall derive and discuss their exact behavior (time development, steady state, linear response) and also some of the approximation methods used for the real problem.

II. SUMMARY OF FORMAL RESULTS OF I

In this section we briefly summarize the formal results of I. We write the Boltzmann equation in the abstract operator form

$$\frac{\partial}{\partial t} f = (D_0 + D_1)f = Df \,, \tag{2.1}$$

where D_0 is an operator representing all the convective terms of the Boltzmann equation, which in the $(\vec{k}\vec{r})$ representation is

$$D_0 = \vec{F}(\vec{r}) \cdot \vec{\nabla}_k + \vec{v}(\vec{k}) \cdot \vec{\nabla}_r \,. \tag{2.2}$$

D_1 is the difference between the in-scattering and out-scattering operators,

$$D_1 = W - w \,. \tag{2.3}$$

For translationally invariant systems, where $\vec{r}$ can be ignored, W in the $\vec{k}$ representation is just the integral operator $\int d^3k' W(\vec{k}, \vec{k}')$ acting on any function of $\vec{k}'$ and w is just the diagonal operator $\int d^3k' w(\vec{k}) \delta(\vec{k} - \vec{k}')$, where

$$w(\vec{k}) = \int d^3k'' W(\vec{k}'', \vec{k}) \,. \tag{2.4}$$

The time development of $f(t)$ is formally given by

$$f(t) = e^{Dt} f(0) \,. \tag{2.5}$$

Various methods of evaluating e^{Dt} were suggested, including straightforward stepwise integration,

$$f((n+1)\Delta t) = e^{D_0 t}(1 + D_1 \Delta t) f(n\Delta t) \quad \text{as} \quad \Delta t \to 0 \,, \tag{2.6}$$

and use of higher-order integration methods on the equation in the "interaction picture,"

$$\frac{\partial}{\partial t} f^I(t) = D_1^I(t) f^I(t) \,, \tag{2.7}$$

where

$$f^I(t) = e^{-D_0 t} f(t) \tag{2.8a}$$

and

$$D_1^I(t) = e^{-D_0 t} D_1 e^{D_0 t} \,. \tag{2.8b}$$

In all such approaches, use is made of the fact that $e^{D_0 t}$ translates all particles along their classical trajectories for a time t without scattering.

A stepwise integration method by Rees[2] appears in this formalism as

$$f(t + 1/\Gamma) = \Gamma \int_0^\infty dt' e^{-\Gamma t'} e^{D_0 t'} (1 + D_1/\Gamma) f(t) \quad \text{as} \quad \Gamma \to \infty \,, \tag{2.9}$$

which is similar to but more complicated than (2.6).

The time-independent problem for the steady-state solution f^s is to be solved by iterating the equation

$$f^s_{n+1} = (U - D_0)^{-1}(D_1 + U) f^s_n \,, \tag{2.10}$$

where U is some operator chosen to optimize the combination of convergence speed and simplicity of evaluation [especially of $(U-D_0)^{-1}$]. Choosing U to be just w, the out-scattering operator, which is diagonal, leads to the iteration schemes of Price[3] and Budd[4] while choosing $U=\Gamma$, a c number, and requiring that $\Gamma \geq w(\vec{k})$ is the method of Rees.

If an additional small perturbation $D'(t)$ is added to D, either by changing the field $\vec{F}$ or the scattering matrix W (and hence w), then we see that the first-order change in f from the steady-state f^s is

$$f'(t)=\int_0^t dt' e^{Dt'} D'(t-t') f^s . \tag{2.11}$$

For a time-independent perturbation, the change in the steady state is

$$f'=\lim_{\eta \to 0+} (\eta - D)^{-1} D' f^s , \tag{2.12}$$

where the operator $(\eta - D)^{-1}$ may be evaluated in any of several ways.

Let us now turn to some models where these calculations can be performed exactly, or at least reduced to quadratures.

III. ONE-BAND MODEL

The first model we wish to consider assumes that the scattering matrix or kernel $W(\vec{k},\vec{k}')$ for scattering from $\vec{k}'$ to $\vec{k}$ is a function only of the final wave vector $\vec{k}$, i.e.,

$$W(\vec{k},\vec{k}')=L(\vec{k}) , \tag{3.1}$$

in which case $w(\vec{k})$ is a constant,

$$w=\int d^3k\, L(\vec{k}) . \tag{3.2}$$

With this choice of W, Boltzmann's equation becomes

$$\left(\frac{\partial}{\partial t} - D_0 + w\right) f(\vec{k}t) = L(\vec{k}) \int d^3k' f(\vec{k}'t) . \tag{3.3}$$

To simplify the subsequent notation, we have assumed that our system is spatially homogeneous and the field $\vec{F}$ is independent of both $\vec{r}$ and $\vec{k}$. Now $\int d^3k' f(\vec{k}'t)$ is independent of time because the scattering process conserves the number of particles. We shall normalize $f(\vec{k}t)$ to the total number of particles:

$$\int d^3k f(\vec{k}t) = N . \tag{3.4}$$

Because the right-hand side of (3.3) does not depend on $f(\vec{k}t)$, the Boltzmann integrodifferential equation becomes just an inhomogeneous differential equation with the formal solution

$$f(\vec{k}t) = e^{-wt} e^{D_0 t} f_i(\vec{k}) + N \int_0^t dt' e^{-wt'} e^{D_0 t'} L(\vec{k}) , \tag{3.5}$$

where $f_i(\vec{k}) = f(\vec{k}, 0)$ is the initial distribution. We recall from I that

$$e^{D_0 t} g(\vec{k}) = g(\vec{k} - \vec{F}t) \tag{3.6}$$

for any function $g(\vec{k})$. Thus,

$$f(\vec{k}t) = e^{-wt} f_i(\vec{k} - \vec{F}t) + N \int_0^t dt' e^{-wt'} L(\vec{k} - \vec{F}t') . \tag{3.7}$$

As $t \to \infty$, $f(\vec{k}t)$ approaches the steady-state solution $f^s(\vec{k})$:

$$f^s(\vec{k}) = \lim_{t\to\infty} f(\vec{k}t) = N \int_0^\infty dt' e^{-wt'} L(\vec{k} - \vec{F}t) . \tag{3.8}$$

We note that in the absence of a field, the equilibrium solution is just

$$f_{eq}(\vec{k}) = N \int_0^\infty dt' e^{-wt'} L(\vec{k}) = NL(\vec{k})/w . \tag{3.9}$$

If the system is also in contact with a heat reservoir in thermal equilibrium at temperature T, then the $\vec{k}$ dependence of $L(\vec{k})$ is specified by

$$L(\vec{k}) \propto e^{-E_k/k_B T} . \tag{3.10}$$

We shall not restrict ourselves to any particular choice for $L(\vec{k})$.

The model scattering we have postulated can be thought of as a two-step process: Particles are destroyed with a probability per unit time w everywhere in $\vec{k}$ space and are then created in a distribution proportional to $L(\vec{k})$. Thus it is identical, in the limit of large systems, to the simultaneous existence of a uniform "sink" distribution of strength w and a source distribution $L(\vec{k})$.

The two terms of the exact solution $f(\vec{k}t)$ can be understood very simply in terms of this physical interpretation of $W(\vec{k},\vec{k}')$. At a time t the particles at $\vec{k}$ are of two kinds, those that have never been scattered and those that have been scattered at least once. The particles of the first kind must have started (at $t=0$) at $\vec{k}-\vec{F}t$. The number surviving without scattering is the original number $f_i(\vec{k}-\vec{F}t)$ times the probability of lasting a time t without scattering, e^{-wt}. The particles of the second kind can be further classified according to the time t' of their last scattering, when they must have been scattered to the point $\vec{k}-\vec{F}(t-t')$. The total number scattered per unit time at any time (including t') to the point $\vec{k}-\vec{F}(t-t')$ is $L(\vec{k}-\vec{F}(t-t'))$, and the fraction of these for which t' is the time of their *last* scattering is $e^{-w(t-t')}$. Since the last scattering time t' can be any time in $(0,t)$, the particles of the second kind contribute the number

$$N \int_0^t dt' e^{-w(t-t')} L(\vec{k} - \vec{F}(t-t'))$$

to the total number at $\vec{k}$ at time t.

Let us now turn to the iterative method for solving the time-independent equation (2.10). We shall consider for U only diagonal operators.

As we have discussed in I, Price has shown that the succession of iterates f_1^s, f_2^s, ... of the equation

$$f_{n+1}^s = (w - D_0)^{-1} W f_n^s \tag{3.11}$$

is simply related to the succession of distributions f_1^b, f_2^b, ... of the particles just before their

respective first, second, etc., scatterings by the scattering operator W. The relation is, in fact,

$$f_n^s = w^{-1} f_n^b \tag{3.12}$$

provided the initial distribution of the time-developing system and the first trial function in the iteration scheme are related by $f_1 = (w - D_0) f_1^s$. Price's analysis, although given on the assumption that W is the actual physical scattering operator, can apply equally well to a generalized scattering operator

$$W_G = W + W_S \ , \tag{3.13}$$

where W_S is some fictitious (but physically allowable) "self-scattering" operator. Thus we see that the succession of iterates of (2.10) are related to the before distributions of a generalized scattering operator

$$W_G = D_1 + U \tag{3.14a}$$

corresponding to a self-scattering operator

$$W_S = U - w \ . \tag{3.14b}$$

If $U = w$, all scatterings are physical.

For the model under consideration, if we take $U = w$, we expect all the before distributions but the first, to show no memory of the initial distribution and, hence, all to be the same. This is obvious from the right-hand side of (3.11), because

$$W f_n^s(\vec{k}) = L(\vec{k}) \int d^3k' f_n^s(\vec{k}') = NL(\vec{k}) \ , \tag{3.15}$$

which is the same for all n. Thus for this model the iteration procedure without self-scattering converges to the steady-state solution in one step!

We can get some insight into Rees's notion of self-scattering by taking $U = \Gamma$, equivalent to a self-scattering $\Gamma - w$, which we shall understand to be diagonal. Price's interpretation of the meaning of the successive iterates f_n^s is still valid, but now the notion of the nth scattering of a particle must be generalized to include both physical scatterings and self-scatterings.

Rees first observed the fact that as $\Gamma \to \infty$, the successive iterates of the time-independent equation are just the time-dependent solution at time intervals $1/\Gamma$. For the model under consideration here, we can show this explicitly and see physically why this should be so. The iterates f_n^s now satisfy the recursion relations

$$f_{n+1}^s = (\Gamma - D_0)^{-1}(W - w + \Gamma) f_n^s \ . \tag{3.16}$$

In view of (3.15),

$$f_{n+1}^s = (\Gamma - D_0)^{-1} NL + (\Gamma - D_0)^{-1}(\Gamma - w) f_n^s \ . \tag{3.17}$$

In terms of the first guess f_1^s, the solution of these equations is

$$f_n^s = -\frac{A^{n-1} - 1}{w - D_0} NL + A^{n-1} f_1^s \ , \tag{3.18}$$

where

$$A = (\Gamma - D_0)^{-1}(\Gamma - w) \ , \tag{3.19}$$

and we have used the fact that $\Gamma - w$ commutes with D_0 (in fact, it is a c number).

Now it is easy to see that for any function $g(\vec{k})$,

$$(w - D_0)^{-1} g(\vec{k}) = \int_0^\infty dt' \, e^{-wt'} e^{D_0 t'} g(\vec{k})$$

$$= \int_0^\infty dt' \, e^{-wt'} g(\vec{k} - \vec{F}t') \tag{3.20a}$$

and

$$\left(\frac{1}{\Gamma - D_0}\right)^n g(\vec{k}) = \frac{1}{(n-1)!}\left(-\frac{\partial}{\partial \Gamma}\right)^{n-1} \frac{1}{\Gamma - D_0} g(\vec{k})$$

$$= \int_0^\infty dt' \, \frac{t'^{n-1}}{(n-1)!} e^{-\Gamma t'} g(\vec{k} - \vec{F}t') \ . \tag{3.20b}$$

Using these relations, we find for the nth iterate:

$$f_n^s(\vec{k}) = -(1 - \Gamma^{-1} w)^{n-1} \int_0^\infty dt' \int_0^\infty dt'' \, p_{n-2}(t') e^{-wt''}$$

$$\times NL(\vec{k} - \vec{F}t' - \vec{F}t'') + \int_0^\infty dt' e^{-wt'} NL(\vec{k} - \vec{F}t')$$

$$+ (1 - \Gamma^{-1} w)^{n-1} \int_0^\infty dt' \, p_{n-2}(t') f_1^s(\vec{k} - \vec{F}t') \ , \tag{3.21}$$

where

$$p_n(t) = \Gamma^{n+1} t^n e^{-\Gamma t}/n! \ . \tag{3.22}$$

The function $p_n(t)$ is identical to the normalized probability density per unit time $\hat{p}_n(t)$ that the nth scattering of a particle occurs at the time t. This is readily seen by observing that if the probability per unit time of a scattering is Γ, then

$$\hat{p}_1(t) = \Gamma e^{-\Gamma t} \tag{3.23a}$$

and

$$\hat{p}_{n+1}(t) = \int_0^t \hat{p}_n(t') \hat{p}_1(t - t') dt' \tag{3.23b}$$

showing that $\hat{p}_n(t) = p_n(t)$, for all n. For large Γ, the probability density function $\hat{p}_n(t)$ is sharply peaked around $t \simeq n/\Gamma$ with an rms deviation $\simeq n^{1/2}/\Gamma$. If for some finite time τ we are interested in $n = \Gamma\tau$, then $p_{\Gamma\tau}(t)$ is peaked around τ with an rms deviation $\tau/(\Gamma\tau)^{1/2} \propto \Gamma^{-1/2}$ as $\Gamma \to \infty$.

Using Price's relation between the successive iterates and the distribution functions before successive collisions, we see that the successive iterates differ from the time-developing solution at the times τ_n because f_n^s is not the distribution function at a single time τ_n, the nth collision times of all the various particles being different. But if $\Gamma \to \infty$, then for a fixed τ, the distribution of the nth

collision times (defining $n=\Gamma\tau$) approaches $\delta(t-\tau)$, *independent of the history of the particles* (the special simplifying feature of this model). Because of this sharpening, the nth before distribution approaches (except for a factor w_G) the time-developing solution at the time $t=n\Gamma$, and therefore the $f_n^s \to f(n\Gamma)$.

To interpret the terms on the right-hand side of (3.21), we also observe that w/Γ is the probability that any given scattering is a physical scattering, so that $(1-\Gamma^{-1}w)^n$ is the probability that in a set of n scatterings none are physical. Letting $\Gamma\to\infty$, $n=\Gamma\tau$, we see that $(1-\Gamma^{-1}w)^n \to e^{-w\tau}$, the probability of no physical scatterings up to the time τ. In the first two terms on the right-hand side of (3.21), if we replace $(1-\Gamma^{-1}w)^{n-1}$ by $e^{-w\tau}$, we may combine them to give

$$N\int_0^\infty dt' e^{-wt'}[1-\int_0^{t'} dt'' p_{n-2}(t'')]L(\vec{k}-\vec{F}t') ,$$

which reduces to the second term on the right-hand side of (3.7) when

$$p_{n-2}(t'') \to \delta(t''-\tau_{n-2}) \tilde{=} \delta(t''-\tau_n) .$$

Let us comment on the effect of the choice of Γ on the convergence of the sequence $f_1^s, f_2^s, \ldots$. When Rees modified the Price integral equation by introducing the possibility of self-scattering, he did it within the context of a stochastic formulation of the problem. As such, Rees naturally excluded the possibility of negative self-scattering probabilities, i.e., he postulated that

$$\Gamma \geq w(\vec{k}) . \tag{3.24}$$

In fact, Rees goes even further by arguing that if this inequality were violated, then the iteration procedure would be unstable, corresponding to an evolution in the negative time direction. This conclusion is wrong, the simple model discussed here being a counter example. We see from (3.21) that the iteration procedure appears to converge whenever

$$|1-\Gamma^{-1}w| < 1, \quad \text{i.e.,} \quad \Gamma > \tfrac{1}{2}w . \tag{3.25}$$

In the range $\frac{1}{2}w < \Gamma < w$, which is disallowed by Rees, the iterations converge but the convergence is oscillatory. In more general problems, where w is a function of $\vec{k}$, the speed of convergence will not be so simply defined, being different in different parts of $\vec{k}$ space. Nevertheless, it is clear from the next model we shall consider that a general improvement in convergence may be achieved by letting $\Gamma - w(\vec{k})$ be negative in some (or even *all*) parts of $\vec{k}$ space. If $w(\vec{k})$ is large in only a small region of $\vec{k}$ space, the relaxation of Rees's constraint can mean a great increase in the convergence speed.

We may conclude our study of the one-band model by considering what happens when the one-band model is modified by a small time-dependent perturbation D'. As we saw in I, the modification f' in the steady-state distribution function can be calculated in at least two different ways:

$$f^{s\prime} = \lim_{\eta\to 0+}\int_0^\infty dt\, e^{-\eta t} e^{Dt} D' f^s$$

$$= \lim_{\eta\to 0+} P_0(\eta) \sum_{m=0}^{\infty} [WP_0(\eta)]^m D' f^s \tag{3.26}$$

and

$$f^{s\prime} = \beta f^s + P_0 \sum_{m=0}^{\infty} (WP_0)^m D' f^s , \tag{3.27a}$$

where

$$\beta = -N^{-1}\int d^3k\, P_0 \sum_{m=0}^{\infty} (WP_0)^m D' f^s . \tag{3.27b}$$

We also saw that consideration of the apparent discrepancy between (3.26) and (3.27) leads to an alternative expression for β:

$$\beta = \langle 1|P_0|r_1\rangle \lim_{\eta\to 0+} \frac{{}_\eta\langle l_1|D'f^s\rangle}{1-\gamma_1(\eta)} , \tag{3.28}$$

where $|r_1\rangle_\eta$ and ${}_\eta\langle l_1|$ are the right and left eigenvectors of $WP_0(\eta)$ belonging to the largest eigenvalue $\gamma_1(\eta)$ and $P_0 = P_0(0)$, $|r_1\rangle = \lim |r_1\rangle_\eta$ as $\eta\to 0+$, etc.

We evaluate these expressions for the one-band model, recalling the important fact that $D'f^s$ has "no particles," i.e.,

$$\int d^3k\, (D'f^s)_k = 0 . \tag{3.29}$$

The straightforward evaluation in (3.26) can be performed if we let $h_0 = D'f^s$ and use the general solution derived in (3.7),

$$h(\vec{k},t) = e^{-wt} h_0(\vec{k}-\vec{F}t) \tag{3.30}$$

because $\int d^3k\, h_0(\vec{k}) = 0$. We have immediately

$$f^{s\prime}(\vec{k}) = \lim_{\eta\to 0+}\int_0^\infty dt e^{\eta t} e^{-wt}\int d^3k' D'(\vec{k}-\vec{F}t,\vec{k}') f^s(\vec{k}') .$$

So long as $w>0$, the limit can be taken inside the integral:

$$f^{s\prime}(\vec{k}) = \int_0^\infty dt\, e^{-\omega t}\int d^3k' D'(\vec{k}-Ft,\vec{k}') f^s(\vec{k}') . \tag{3.31}$$

The series method (3.27) proposed by Rees[5] can also be applied because

$$WP_0 D' f^s = L(\vec{k})\int d^3k' \int_0^\infty dt\, e^{-wt}(D'f^s)_{k'-Ft} , \tag{3.32}$$

because $\int d^3k' D'(\vec{k}',\vec{k}'') = 0$. Thus only the $m=0$ term in $\sum_0^\infty (WP_0)^m D' f^s$ contributes, a fact related, of course, to the one-step convergence of the iteration procedure for f^s in the unperturbed problem. We have

$$f^{s\prime}(\vec{k}) = \beta f^s(\vec{k}) + \int_0^\infty dt\, e^{-wt}(D'f^s)_{k-Ft} \,. \qquad (3.33)$$

Similarly

$$\beta = -\int d^3k \int_0^\infty dt\, e^{-wt}(D'f^s)_{k-Ft} = 0 \,. \qquad (3.34)$$

Thus for this particular model, Rees's assertion[5] that $\beta = 0$ is true. The alternative expression (3.28) also gives zero because ${}_\eta\langle l_1|$ does not depend on η while $\gamma_1 = w/(w+\eta)$, so that

$$\lim_{\eta\to 0+} \frac{{}_\eta\langle l_1|D'f^s\rangle}{1-\gamma_1(\eta)} = \lim_{\eta\to 0+} \frac{0}{\eta(w+\eta)} = 0 \,.$$

Because this model is too primitive to exemplify some of the subtleties of the perturbation theory discussed in I, let us turn to a model that is more illustrative.

IV. TWO-BAND MODEL

In Sec. III we considered a model in which each collision wiped out all previous memory. The only role of the convective term was to describe what happened since the previous collision.

In this section we consider a more general model in which the memory of certain gross features persists through collisions, although the convective term still plays a secondary role in determining the time evolution and the steady state.

The model we wish to consider can be thought of as a two-band generalization of the model of Sec. III. We assume that the one-electron states can be characterized by a wave vector $\vec{k}$ and a band index α. We use the word "band" because we shall assume that all classical orbits lie wholly within one band or the other. The model is characterized by assuming that the scattering probability into the state $\vec{k}$ and band α depends on the band α', but not the wave vector $\vec{k}'$ of the initial state.[6] In the $\vec{k}\alpha$ representation, the two-band model is thus characterized by a scattering operator having the structure

$$W(\vec{k}\alpha, \vec{k}'\alpha') = L_{\alpha\alpha'}(\vec{k}) \,. \qquad (4.1)$$

The out-scattering operator is the diagonal operator

$$w_\alpha = \sum_{\alpha'} w_{\alpha'\alpha} \,, \qquad (4.2)$$

where

$$w_{\alpha'\alpha} = \int d^3k'\, W(\vec{k}'\alpha', \vec{k}\alpha) = \int d^3k'\, L_{\alpha'\alpha}(\vec{k}') \,. \qquad (4.3)$$

With this scattering operator, Boltzmann's equation decomposes into two-coupled integrodifferential equations

$$\left(\frac{\partial}{\partial t} - \vec{F}\cdot\vec{\nabla}_k + w_1\right) f(\vec{k}1, t) = L_{11}(\vec{k})N(1,t) + L_{12}(\vec{k})N(2,t) \,, \qquad (4.4)$$

$$\left(\frac{\partial}{\partial t} - \vec{F}\cdot\vec{\nabla}_k + w_2\right) f(\vec{k}2, t) = L_{21}(\vec{k})N(1,t) + L_{22}(\vec{k})N(2,t) \,,$$

where

$$N(\alpha, t) = \int d^3k\, f(\vec{k}\alpha, t) \,. \qquad (4.5)$$

Let us first consider the exact time evolution toward the steady state. In the course of time, two tendencies are occurring in this model: The total populations of the two bands are changing and approaching a steady state, and within each band there is a continual tendency toward an internal steady state appropriate to the instantaneous population of the band. The second of these tendencies is just the behavior of the one-band model and depends on the applied field in a trivial way; the first tendency is independent of field because orbit motion (the only manifestation of the field) does not change the band populations.

To separate these two mechanisms, we may integrate Eqs. (4.4) over $\vec{k}$. We obtain coupled equations for $N(1,t)$ and $N(2,t)$ which we write in matrix form:

$$\left[\underline{1}\,\frac{\partial}{\partial t} + \begin{pmatrix} w_{21} & -w_{12} \\ -w_{21} & w_{12} \end{pmatrix}\right]\begin{pmatrix} N(1,t) \\ N(2,t) \end{pmatrix} = 0 \,. \qquad (4.6)$$

The formal solution of these equations is

$$\begin{pmatrix} N(1,t) \\ N(2,t) \end{pmatrix} = \exp\left[-\begin{pmatrix} w_{21} & -w_{12} \\ -w_{21} & w_{12} \end{pmatrix} t\right]\begin{pmatrix} N(1,0) \\ N(2,0) \end{pmatrix} \,. \qquad (4.7)$$

The exponential operator on the right-hand side can be evaluated if one knows the eigenvalues λ_1 and λ_2, the corresponding right eigenvectors $\underline{\phi}_1$ and $\underline{\phi}_2$, and the corresponding left eigenvectors $\underline{\psi}_1$ and $\underline{\psi}_2$ of the matrix

$$\begin{pmatrix} w_{21} & -w_{12} \\ -w_{21} & w_{12} \end{pmatrix} \,.$$

A direct solution of the eigenvalue problem yields

$$\lambda_1 = 0, \quad \underline{\phi}_1 = \begin{pmatrix} w_{12} \\ w_{21} \end{pmatrix}, \quad \underline{\psi}_1 = \begin{pmatrix} 1 & 1 \end{pmatrix} ; \qquad (4.8a)$$

$$\lambda_2 = w_{12} + w_{21} \equiv \lambda, \quad \underline{\phi}_2 = \begin{pmatrix} 1/\lambda \\ -1/\lambda \end{pmatrix}, \quad \underline{\psi}_2 = (\omega_{21} - \omega_{12}) \,. \qquad (4.8b)$$

In terms of these, we have the solution to (4.6):

$$\underline{N}(t) = \underline{\phi}_1\underline{\psi}_1\cdot\underline{N}(0) + e^{-\lambda t}\underline{\phi}_2\underline{\psi}_2\cdot\underline{N}(0)$$

or explicitly

$$\begin{pmatrix} N(1,t) \\ N(2,t) \end{pmatrix} = \frac{1}{\lambda}\begin{pmatrix} w_{12} + w_{21}e^{-\lambda t} & w_{12}(1-e^{-\lambda t}) \\ w_{21}(1-e^{-\lambda t}) & w_{21} + w_{12}e^{-\lambda t} \end{pmatrix}\begin{pmatrix} N(1,0) \\ N(2,0) \end{pmatrix} \,. \qquad (4.9)$$

We see that as time goes on the populations of the two bands change and ultimately approach steady-state values:

$$\begin{aligned} N(1,t) &\to N^s(1) = (w_{12}/\lambda) \,, \\ N(2,t) &\to N^s(2) = (w_{21}/\lambda)N \,. \end{aligned} \qquad (4.10)$$

The solution $f(\vec{k}\alpha t)$ of (4.4) can now be obtained formally, since the right-hand sides of (4.4) can be considered as a known function of t. Thus, $f(\vec{k}\alpha, t)$ satisfies the equation

$$\left(\frac{\partial}{\partial t} - D_0 + w_\alpha\right) f(\vec{k}\alpha, t) = g(\vec{k}\alpha, t) , \tag{4.11}$$

where

$$g(\vec{k}\alpha, t) = L_{\alpha 1}(\vec{k})N(1, t) + L_{\alpha 2}(\vec{k})N(2, t) . \tag{4.12}$$

One has the solution

$$f(\vec{k}\alpha, t) = e^{-w_\alpha t} f(\vec{k} - \vec{F}t, \alpha, 0) + \int_0^t dt' e^{-w_\alpha t'} g(\vec{k} - \vec{F}t', \alpha, t - t') . \tag{4.13}$$

When $t \to \infty$, the distribution approaches the steady-state distribution

$$f^s(\vec{k}\alpha) = \int_0^\infty dt' e^{-w_\alpha t'} \sum_{\alpha'} L_{\alpha\alpha'}(\vec{k} - \vec{F}t')N^s(\alpha') , \tag{4.14}$$

which satisfies the steady-state equation

$$(-\vec{F}\cdot\vec{\nabla}_k + w_\alpha) f^s(\vec{k}\alpha) = \sum_{\alpha'} L_{\alpha\alpha'}(\vec{k})N^s(\alpha') . \tag{4.15}$$

Let us now return to the iterative approach to the steady-state equation, including at the same time the possibility of self-scattering as proposed by Rees. We wish to see how the choice of self-scattering affects the convergence of the iterations. We start from the iterative form of the steady-state equations given in (3.16). Explicitly, these are (omitting the superscripts henceforth)

$$f_{n+1}(\vec{k}\alpha) = (\Gamma - D_0)^{-1}\left(\sum_{\alpha'} L_{\alpha\alpha'}(\vec{k})N_n(\alpha') + (\Gamma - w_\alpha) f_n(\vec{k}\alpha)\right) , \tag{4.16}$$

where

$$N_n(\alpha) = \int d^3k\, f_n(\vec{k}\alpha) . \tag{4.17}$$

If we integrate over $\vec{k}$, we obtain a recursion relation for the band population functions:

$$N_{n+1}(\alpha) = \sum_{\alpha'} \int d^3k\, (\Gamma - D_0)^{-1} L_{\alpha\alpha'}(\vec{k})N_n(\alpha') + (\Gamma - w_\alpha) \int d^3k\, (\Gamma - D_0)^{-1} f_n(\vec{k}\alpha) . \tag{4.18}$$

Introducing the integral representation

$$(\Gamma - D_0)^{-1} = \int_0^\infty dt\, e^{-\Gamma t} e^{D_0 t} , \tag{4.19}$$

we have that

$$\int d^3k\, (\Gamma - D_0)^{-1} L_{\alpha\alpha'}(\vec{k}) = w_{\alpha\alpha'}/\Gamma \tag{4.20}$$

and

$$\int d^3k\, (\Gamma - D_0)^{-1} f_n(\vec{k}\alpha) = N_n(\alpha)/\Gamma , \tag{4.21}$$

so we obtain the auxiliary recursion relations for the $N_n(\alpha)$;

$$\begin{pmatrix} N_{n+1}(1) \\ N_{n+1}(2) \end{pmatrix} = \underline{B}\cdot\begin{pmatrix} N_n(1) \\ N_n(2) \end{pmatrix} = \underline{B}^n\cdot\begin{pmatrix} N_1(1) \\ N_1(2) \end{pmatrix} , \tag{4.22}$$

where

$$\underline{B} = \begin{pmatrix} 1 - w_{21}/\Gamma & w_{12}/\Gamma \\ w_{21}/\Gamma & 1 - w_{12}/\Gamma \end{pmatrix} . \tag{4.23}$$

If we expand $\underline{B}$ in its left and right eigenvectors given in Eqs. (4.8), it is elementary to evaluate $\underline{B}^n$. One obtains the solution of (4.22):

$$N_{n+1}(\alpha) = \frac{w_{\alpha\bar{\alpha}}}{\lambda} N - \left(1 - \frac{\lambda}{\Gamma}\right)^n \left(\frac{w_{\alpha\bar{\alpha}}}{\lambda} N_1(\bar{\alpha}) - \frac{w_{\bar{\alpha}\alpha}}{\lambda} N_1(\alpha)\right) , \tag{4.24}$$

where $\bar{\alpha}$ is the opposite index to α. We see that to achieve most rapid convergence of the band populations to their steady-state values, one should choose $\Gamma = w_{12} + w_{21}$, although the iterations will converge provided only that $\Gamma > \frac{1}{2}(w_{12} + w_{21})$.

Now it is true that convergence of the auxiliary equations is not equivalent to over-all convergence. Choosing $\Gamma = w_{12} + w_{21}$ may even lead to divergence in the iterations within each band. For this particular model, the approach to steady state within each band and in the relative band populations are three distinct processes that in a sense compete with one another. The choice of a single Γ must be made to speed the convergence of as many of these three processes as possible without paying too large a price (like divergence) in any one of them. To see this explicitly, let us evaluate the successive iterates f_1, f_2, etc.

Let us henceforth suppress all $\vec{k}$ dependences and represent all functions of one- and two-band indices by two-component vectors (like $\underline{f}$) and 2×2 matrices (like $\underline{L}$). Multiples of the 2×2 unit matrix will be written simply as scalars. Then Eq. (4.16) can be rewritten

$$\underline{f}_{n+1} = (\Gamma - D_0)^{-1}[\underline{L}\cdot\underline{N}_n + (\Gamma - \underline{w})\cdot\underline{f}_n] . \tag{4.25}$$

From (4.22) in this compressed notation,

$$\underline{N}_n = \underline{B}^{n-1}\cdot\underline{N}_1 , \quad \text{all } n \geq 1 . \tag{4.26}$$

The formal solution of (4.25), after substituting (4.26), is

$$\underline{f}_{n+1} = \underline{A}^n\cdot\underline{f}_1 + \left(\sum_{r=0}^{n-1} \underline{A}^r\cdot(\Gamma - D_0)^{-1}\underline{L}\cdot\underline{B}^{n-1-r}\right)\cdot\underline{N}_1 , \tag{4.27}$$

where

$$\underline{A} = (\Gamma - D_0)^{-1}(\Gamma - \underline{w}) . \tag{4.28}$$

Using the fact that the two factors in $\underline{A}$ commute for this model, we can simplify (4.28) as

$$\underline{f}_{n+1} = \left(1 - \frac{\underline{w}}{\Gamma}\right)^{n} \cdot \left(1 - \frac{D_0}{\Gamma}\right)^{-n} \underline{f}_1 + \Gamma \sum_{r=0}^{n-1} \left(1 - \frac{D_0}{\Gamma}\right)^{-(n+1)} \times \left(1 - \frac{\underline{w}}{\Gamma}\right)^{r} \cdot \underline{L} \cdot \underline{B}^{n-1-r} \cdot \underline{N}_1 \,. \tag{4.29}$$

Now the operator $(1 - D_0/\Gamma)^{-1}$ is a smoothing operator in the sense that

$$(1 - D_0/\Gamma)^{-1}\chi(\vec{k}) = \Gamma \int_0^\infty dt\, e^{-\Gamma t}\chi(\vec{k} - \vec{F}t) \tag{4.30}$$

is a somewhat smoother function than $\chi(\vec{k})$ but with an unchanged normalization

$$\int d^3k\,(1 - D_0/\Gamma)^{-1}\chi(\vec{k}) = \int d^3k\,\chi(\vec{k}) \,. \tag{4.31}$$

In (4.29), the first term on the right-hand side will go to zero with increasing n provided that

$$|1 - w_\alpha/\Gamma| < 1,$$

i.e.,

$$\Gamma > \tfrac{1}{2}w_1 \text{ and } \Gamma > \tfrac{1}{2}w_2 \,. \tag{4.32}$$

In the sum of (4.29), a typical term, aside from the smoothing operator $(1 - D_0/\Gamma)^{-(n+1)}$ is

$$\left(1 - \frac{\underline{w}}{\Gamma}\right)^{r} \cdot \underline{L} \cdot \underline{B}^{n-1-r} \cdot \underline{N}_1 = \sum_{\alpha\alpha'} \left[\left(1 - \frac{w_\alpha}{\Gamma}\right)^{r} L_{\alpha\alpha'}\phi_1(\alpha')(\underline{\psi}_1 \cdot \underline{N}_1) + \left(1 - \frac{w_\alpha}{\Gamma}\right)^{r}\left(1 - \frac{\lambda}{\Gamma}\right)^{n-1-r} L_{\alpha\alpha'}\phi_2(\alpha')(\underline{\psi}_2 \cdot \underline{N}_2)\right] \,. \tag{4.33}$$

We see that convergence is assured if

$$\Gamma > \tfrac{1}{2}w_1, \quad \Gamma > \tfrac{1}{2}w_2, \quad \text{and} \quad \Gamma > \tfrac{1}{2}\lambda = \tfrac{1}{2}(w_{12} + w_{21}) \,. \tag{4.34}$$

To speed the convergence, Γ should be chosen to minimize, in some sense, the quantities $|1 - w_1/\Gamma|$, $|1 - w_2/\Gamma|$, and $|1 - \lambda/\Gamma|$. For certain relative magnitudes of w_{11}, w_{22}, w_{12}, and w_{21}, the optimal choice for Γ, while never completely unambiguous, may nevertheless be quite surprising. For example, if $w_1 \gtrsim w_2 \gg \lambda$, the choice of $\Gamma \simeq w_1$ or w_2 leaves the factor $|1 - \lambda/\Gamma|$ too near unity. A better choice would be some Γ in the interval $(\frac{1}{2}w_1, w_2)$. This result is to be contrasted with the condition set down by Rees that $\Gamma - w$ must be nonnegative which, for this model, becomes the two Rees conditions $\Gamma \geq w_1$ and $\Gamma \geq w_2$. *Thus for* $w_1 > w_2 \gg \lambda$, *the optimal* Γ *will violate the Rees condition everywhere.*

Let us now consider a small time-independent perturbation D' applied to the two-band model. The considerations presented in I are now nontrivial and can be exhibited explicitly.

We first consider the method of direct time integration (3.26). If we define the "particle-less state" h_0 by

$$h_0(\vec{k}, \alpha) = \langle \vec{k}\alpha |\, D' f^s \rangle \tag{4.35}$$

and the corresponding band population

$$M_0(\alpha) = \int d^3k\, h_0(\vec{k}, \alpha) \,, \tag{4.36}$$

then the time-developing solution to the Boltzmann equation satisfying the initial condition $h(\vec{k}\alpha, 0) = h_0(\vec{k}, \alpha)$ can be deduced from (4.12) and (4.13):

$$h(\vec{k}\alpha, t) = e^{-w_\alpha t}\left[h_0(\vec{k} - \vec{F}t, \alpha) + \lambda^{-1}\sum_{\alpha'} L_{\alpha\alpha'}(\vec{k} - \vec{F}t)M_0(\alpha')\right] \,. \tag{4.37}$$

We have used the solution to the auxiliary equation for $M(\alpha, t)$:

$$M(\alpha, t) = e^{-\lambda t}M_0(\alpha) \,, \tag{4.38}$$

which can be deduced from (4.9) if we recall that $\sum_\alpha M_0(\alpha) = 0$. In terms of $h(\vec{k}\alpha, t)$ and $M(\alpha, t)$, the corrections to the steady-state distribution function and band population are simply

$$f^{s\prime}(\vec{k}, \alpha) = \lim_{\eta \to 0+} \int_0^\infty dt\, e^{-\eta t} h(\vec{k}\alpha, t) \,, \tag{4.39}$$

$$N^{s\prime}(\alpha) = \lim_{\eta \to 0+} \int_0^\infty dt\, e^{-\eta t} M(\alpha, t) = \lambda^{-1}M_0(\alpha) \,. \tag{4.40}$$

The time integration in (4.37) can be performed, in principle, if the specific forms for $h_0(\vec{k}, \alpha)$ and $L_{\alpha\alpha'}(\vec{k})$ are known.

Let us now see what is involved in deriving the same results using the series method (3.27a) together with the two-band generalization of (3.27b):

$$\beta = -N^{-1}\sum_\alpha \int d^3k \left(P_0 \sum_{m=0}^{\infty} (WP_0)^m\right)_{\vec{k},\alpha} \,. \tag{4.41}$$

Because of the special structure of W for the two-band model, it is easy to show that, for $m \geq 1$,

$$[(Wp_0)^m h_0]_{\vec{k},\alpha} = \underline{L}(\vec{k}) \cdot \underline{w}^{-1} \cdot (\underline{\overline{w}} \cdot \underline{w}^{-1})^{m-1} \cdot \underline{M}_0 \,. \tag{4.42}$$

We have introduced the matrix

$$(\underline{\overline{w}})_{\alpha\alpha'} = w_{\alpha\alpha'} \tag{4.43}$$

which is not to be confused with the diagonal matrix $(\underline{w})_{\alpha\alpha'} = w_\alpha \delta_{\alpha\alpha'}$. Thus

$$P_0 \sum_{m=0}^{\infty} (WP_0)^m h_0 = (\underline{w} - D_0)^{-1} \cdot \left(\underline{h}_0(\vec{k}) + \underline{L}(\vec{k}) \cdot \underline{w}^{-1} \cdot \sum_{m=0}^{\infty} (\underline{\overline{w}} \cdot \underline{w}^{-1})^m \cdot \underline{M}_0\right) \tag{4.44}$$

and, by (4.41),

$$\beta = -N^{-1}\sum_{\alpha} w_{\alpha}^{-1}\left(\sum_{m=0}^{\infty}(\underline{\bar{w}}\cdot\underline{w}^{-1})^{m}\cdot\underline{M}_{0}\right)_{\alpha} . \qquad (4.45)$$

To evaluate $\sum_0^\infty (\underline{\bar{w}}\cdot\underline{w}^{-1})^m\cdot\underline{M}_0$, it is useful to express the 2×2 matrix $\underline{\bar{w}}\cdot\underline{w}^{-1}$ in terms of its right and left eigenvectors. The two eigenvalues are

$$\mu_1 = 1$$

and

$$\mu_2 = (w_{11}/w_1) + (w_{22}/w_2) - 1 . \qquad (4.46)$$

Because the left eigenvector belonging to μ_1 is (1 1), which is orthogonal to $\underline{M}_0$, the contribution of every term of this series arising from the first eigenvalue and its eigenvectors is zero. The second eigenvalue and its eigenvectors can be shown to give

$$\sum_{m=0}^{\infty}(\underline{\bar{w}}\cdot\underline{w}^{-1})^{m}\cdot\underline{M}_{0} = M_{0}(1)\left(\sum_{m=0}^{\infty}\mu_{2}^{m}\right)\begin{pmatrix}1\\-1\end{pmatrix} . \qquad (4.47)$$

Because $|\mu_2|<1$, the series converges to $(1-\mu_2)^{-1}$ and its truncation after a few terms may be a reasonable approximation.

We can also compute β by combining (4.45) and (4.47), obtaining

$$\beta = -\frac{1}{N}\left(\frac{1}{w_1}-\frac{1}{w_2}\right)\frac{M_0(1)}{1-\mu_2} = \frac{(w_1-w_2)M_0(1)}{w_{21}w_2+w_{12}w_1} . \qquad (4.48)$$

Using the integral representation

$$P_0 = \int_0^\infty dt\, e^{-w_\alpha t}e^{D_0 t} \qquad (4.49)$$

and the explicit steady-state solution

$$f^s(\vec{k},\alpha) = \int_0^\infty dt\, e^{-w_\alpha t}\sum_{\alpha'}L_{\alpha\alpha'}(\vec{k}-\vec{F}t)N^s(\alpha') \quad (4.14)$$

as well as (4.44), (4.47), and (4.48), one can easily verify that the expression

$$\beta f^s + P_0\sum_{m=0}^{\infty}(WP_0)^m h_0$$

yields the same steady-state correction as obtained by direct time integration from (4.37) and (4.39):

$$\underline{f}^{s\prime}(\vec{k}) = \int_0^\infty dt\, e^{-\underline{w}t}\cdot\underline{h}_0(\vec{k}-\vec{F}t) + \lambda^{-1}\int_0^\infty dt\, e^{-\underline{w}t}\cdot\underline{L}(\vec{k}-\vec{F}t)\cdot\underline{M}_0 . \qquad (4.50)$$

It is also interesting to see that the expression (3.28) for β obtained by a careful consideration of the limit $\eta\to0+$ yields the same result as we obtained in (4.48). The eigenvalue problems for the left and right eigenvectors are

$$WP_0(\eta)|r_1\rangle_\eta = \gamma_1(\eta)|r_1\rangle_\eta , \qquad (4.51a)$$

$${}_\eta\langle l_1|WP_0(\eta) = \gamma_1(\eta)\,{}_\eta\langle l_1| . \qquad (4.51b)$$

These problems can be solved by integrating over $\vec{k}$ space to derive two auxiliary eigenvalue problems for two-component eigenvectors:

$$\underline{\bar{w}}\cdot(\eta+\underline{w})^{-1}\cdot\underline{V}_1(\eta) = \gamma_1(\eta)\underline{V}_1(\eta) , \qquad (4.52a)$$

$$\underline{U}_1(\eta)\cdot\underline{\bar{w}}\cdot(\eta+\underline{w})^{-1} = \gamma_1(\eta)\underline{U}_1(\eta) , \qquad (4.52b)$$

where

$$V_1(\alpha;\eta) = \int d^3k\, r_1(\vec{k}\alpha;\eta) , \qquad (4.53a)$$

$$U_1(\alpha;\eta) = \int d^3k\, l_1(\vec{k}\alpha;\eta) . \qquad (4.53b)$$

In terms of these auxiliary eigenvectors

$$\underline{r}_1(\vec{k};\eta) = \frac{1}{\gamma_1(\eta)}\underline{L}(\vec{k})\cdot\frac{1}{\eta+\underline{w}}\cdot\underline{V}_1(\eta) , \qquad (4.54a)$$

$$\underline{l}_1(\vec{k};\eta) = \underline{U}_1(\eta), \text{ independent of } \vec{k} . \qquad (4.54b)$$

We have assumed that $\vec{k}$ space has unit volume.

Retaining only the lowest necessary orders in η, one finds that

$$\underline{U}(\eta) = (1\ 1) - \eta\frac{w_1}{w_{21}}\left(\frac{1}{w_1}\ \frac{\lambda}{w_{21}w_2+w_{12}w_1}\right) , \qquad (4.55)$$

so that

$${}_\eta\langle l_1|\,h_0\rangle = \eta\frac{w_1-w_2}{w_{21}w_2+w_{12}w_1}M_0(1) . \qquad (4.56)$$

One also finds that

$$\underline{V}_1(0) = \frac{1}{1-\mu_2}\begin{pmatrix}w_{12}/w_2\\ w_{21}/w_1\end{pmatrix} , \qquad (4.57)$$

so that

$$\langle 1|P_0|\,r_1\rangle_0 = \sum_\alpha\frac{1}{w_\alpha}V_1(\alpha;0) = \frac{1}{1-\mu_2}\frac{\lambda}{w_1w_2} . \qquad (4.58)$$

Finally

$$1-\gamma_1(\eta) = \eta\frac{\lambda}{1-\mu_2}\frac{1}{w_1w_2} . \qquad (4.59)$$

Substituting (4.57)–(4.59) into (3.28) for β, we obtain

$$\beta = \frac{(w_1-w_2)M_0(1)}{w_{21}w_2+w_{12}w_1}$$

agreeing exactly with our previous result (4.48). Thus, as we explained in I, the inner product ${}_\eta\langle l_1|\,h_0\rangle$ is of order η and the series $\sum_0^\infty\mu_1^m$ is of order η^{-1}. The correct result for β is obtained if the ratio is taken *before* the limit $\eta\to0+$.

V. SUMMARY

We have presented two models for which the Boltzmann equation representing a hot-electron problem can be solved exactly. The models are used to illustrate the iterative approach to the steady state introduced by Price and the modifications involving the notion of "self-scattering" introduced by Rees. From these illustrations it is immediate-

ly apparent why the Rees iterations yield the exact time evolution in the limit of an infinitely large self-scattering rate. It is also demonstrated that the notion of self-scattering can be generalized to allow for negative self-scattering rates, in direct violation of a condition imposed by Rees. It is even shown for one model that for certain choices of the model parameters, the optimum self-scattering rate may be negative everywhere.

The models are also used to discuss various methods of evaluating the first-order corrections to the steady state resulting from a small perturbation in the system. It is shown that the summations arising must be treated carefully, and consistently, lest the results obtained be simply incorrect.

The advantages of the models for the evaluation of numerical and analytic approximation procedures used in calculating other hot-electron properties, for which the models will usually yield exact solutions, need hardly be stressed.

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[6]In some ways this is a model for any physical situation where there are two valleys in $\vec{k}$ space, sufficiently removed from one another that one can neglect the orbits that link them. There is, unfortunately, one important respect in which the interesting physical situations differ from the models. In the physical situations, the *intervalley* scattering rate varies dramatically with position within the initial valley.

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Thermal and Electrical Transport in a Tungsten Crystal for Strong Magnetic Fields and Low Temperatures*

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Direct-current electrical and thermal transport coefficients were experimentally determined in a very pure tungsten crystal at six temperatures in the range 1.4–4.1 K and at magnetic fields up to 22 kOe. The field was applied along a [100] axis and each of the coefficients measured along an equivalent ⟨100⟩ direction. Kinetic coefficients computed from the data were interpreted in terms of a Sondheimer-Wilson–type multiband relaxation-time model. The results were generally consistent with the extensive literature on the fermiology of tungsten. Galvanomagnetic data were approximately independent of temperature, a result implying elastic scattering and a common relaxation time for all transport effects, but the field dependence and magnitude of the thermal and Righi-Leduc resistivities were both distinctly less than those predicted by the Wiedemann-Franz law when a reasonable value of the lattice conductivity was assumed. A density of states computed from the Nernst-Ettingshausen coefficient was consistent with values reported from specific-heat measurements, but displayed an anomalous temperature dependence similar to that of the Righi-Leduc coefficient.

I. INTRODUCTION

The temperature and magnetic field dependence of six galvanomagnetic and thermomagnetic transport coefficients of a tungsten monocrystal are reported here. The dc measurements were performed at liquid-helium-4 temperatures in magnetic fields up to 22 kOe directed along a [100] axis transverse to the plane of the effects in the body-centered-cubic (bcc) crystal.

Provided the magnetic field is applied along an axis of three-, four-, or sixfold rotational symmetry, and the electrical and thermal fluxes are constrained to the plane normal to that axis, only six kinetic transport coefficients are required to determine all of the thermogalvanomagnetic phenomena in a metallic crystal. The transverse-even effects vanish, and the six kinetic coefficients are calculable in terms of only six measurable nonkinetic coefficients. This is a consequence of the Onsager relations.[1,2]

A study of transport phenomena is motivated by two distinct, but coupled, goals. First, one would like to use the transport effects as a tool for investigating the structure and dynamics of the electron–lattice-defect system in a class of materials. Second, one wishes to understand mechanisms present in the transport process and to determine